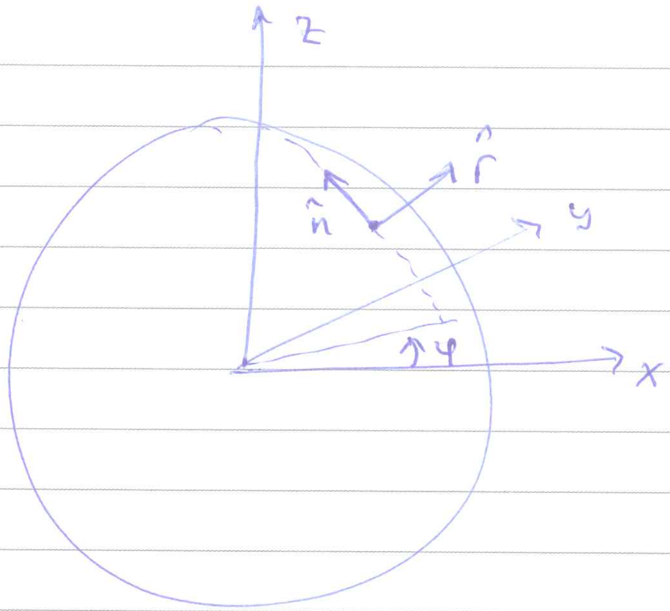
$x \times y = z$

$y \times z = x$

$z \times x = y$



θ = lattitude (north) of equator

φ = longitude (east of Greenwich)

$$\hat{r} = \cos\varphi \cos\theta\, \hat{x} + \sin\varphi \cos\theta\, \hat{y} + \sin\theta\, \hat{z}$$

$$\hat{n} = \text{north} = -\cos\varphi \sin\theta\, \hat{x} - \sin\varphi \sin\theta\, \hat{y} + \cos\theta\, \hat{z}$$

$$\hat{e} = \text{east} = -\hat{r} \times \hat{n} = \hat{n} \times \hat{r}$$

$$\hat{e} = \hat{z}\left(-\sin\varphi \cos\varphi \sin\theta \cos\theta + \sin\varphi \cos\varphi \sin\theta \cos\theta\right)$$
$$+ \hat{x}\left(-\sin\varphi \sin^2\theta - \sin\varphi \cos^2\theta\right)$$
$$+ \hat{y}\left(\cos^2\theta \cos\varphi + \cos\varphi \sin^2\theta\right)$$

$$\hat{e} = -\sin\varphi\, \hat{x} + \cos\varphi\, \hat{y}$$

$$A^{ab} = -\frac{1}{3}\cos^2 i\; u^a u^b + \frac{2}{3}\cos^2 i\; v^a v^b .$$

$$B^{ab} = \frac{2}{3} u^a u^b + \frac{2}{3} v^a v^b$$

$$C^{ab} = \cos i\; u^a v^b + v^a u^b$$

$$\mathcal{M}_{jk} = m_{jk} - \frac{1}{3} g_{jk} m^l{}_l$$

$$h^{TT\,ij} = \frac{2}{r}\left(p^i{}_k p^j{}_l \ddot{\mathcal{M}}^{kl} + \frac{1}{2} p^{ij} \ddot{\mathcal{M}}_{kl} n^k n^l\right)$$

$$p_{ij} = \delta_{ij} - n_i n_j$$

$$h^{TT\,ij} = \frac{2}{R}\Big(p^i{}_k p^j{}_l \left(\ddot{m}^{kl} - \frac{1}{3} g^{kl} \ddot{m}^s{}_s\right)$$
$$+ \frac{1}{2} p^{ij}\left(\ddot{m}_{kl} n^k n^l - \frac{1}{3} g_{kl} n^k n^l \ddot{m}^s{}_s\right)\Big)$$

$$= \frac{2}{R}\Big(p^i{}_k p^j{}_l \ddot{m}^{kl} - \frac{1}{3} p^{ij} \ddot{m}^s{}_s + \frac{1}{2} p^{ij} \ddot{m}_{kl}\left(\delta^{kl} - p^{kl}\right)$$
$$- \frac{1}{6} p^{ij} \ddot{m}^s{}_s\Big)$$

$$= \frac{2}{R}\Big(p^i{}_k p^j{}_l \ddot{m}^{kl} - \cancel{\frac{1}{2} p^{ij} \ddot{m}^s{}_s} + \cancel{\frac{1}{2} p^{ij} \ddot{m}^s{}_s}$$
$$- \frac{1}{2} p^{ij} \ddot{m}_{kl} p^{kl}\Big)$$

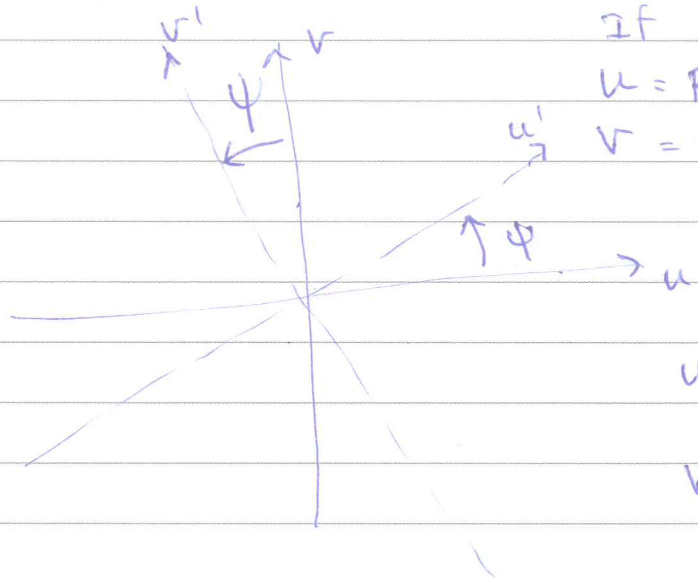
$$= \frac{2}{R}\left(p^i{}_k p^j{}_l \ddot{m}^{kl} - \frac{1}{2} p^{ij} \ddot{m}_{kl} p^{kl}\right)$$

(1) Mass quadrupole, work in 2-d projective plane

$$m^{ij} = \int \rho \, x^i x^j \, d^3x$$

In u, v plane

$$m^{ij} = \begin{bmatrix} u^2 & uv \\ vu & v^2 \end{bmatrix}$$



If $\psi = 0$ then

$u = R \cos \omega t$

$v = R \cos i \sin \omega t$

$$u' = \cos\psi \, u + \sin\psi \, v$$

$$v' = -\sin\psi \, u + \cos\psi \, v$$

$$\begin{matrix} u \\ v \end{matrix} = \begin{matrix} \cos\psi \, u' - \sin\psi \, v' \\ \sin\psi \, u' + \cos\psi \, v' \end{matrix}$$

Trace is $u^2 + v^2$, so traceless is

$$\begin{bmatrix} \frac{1}{2}(u^2 - v^2) & uv \\ uv & \frac{1}{2}(v^2 - u^2) \end{bmatrix}$$

$\sin 2x = 2\sin x \cos x$

$\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

$u' = R \cos i \cos \omega t$

$v' = R \sin \omega t$

$$u = R\left[\cos\psi\,(\cos i \cos\omega t) - \sin\psi \sin\omega t\right]$$

$$v = R\left[\sin\psi \cos i \cos\omega t + \cos\psi \sin\omega t\right]$$

m^{ij} projected, traceless is as follows. Set $R=1$

$$u^2 = \cos^2\psi \cos^2 i \cos^2\omega t + \sin^2\psi \sin^2\omega t - 2\sin\psi\cos\psi\cos i \sin\omega t \cos\omega t$$

$$v^2 = \sin^2\psi \cos^2 i \cos^2\omega t + \cos^2\psi \sin^2\omega t + 2\sin\psi\cos\psi\cos i \sin\omega t \cos\omega t$$

$$u^2 - v^2 = (\cos^2\psi - \sin^2\psi)\cos^2 i \cos^2\omega t + (\sin^2\psi - \cos^2\psi)\sin^2\omega t - 4\sin\psi\cos\psi\cos i \sin\omega t\cos\omega t$$

Note $\cos^2\psi - \sin^2\psi = \cos 2\psi$

$$\boxed{u^2 - v^2 = \cos 2\psi\left[\cos^2 i \cos^2\omega t - \sin^2\omega t\right] - 2\sin 2\psi \cos i \sin\omega t \cos\omega t}$$

$$uv = \sin\psi\cos\psi\left(\cos^2 i \cos^2\omega t - \sin^2\omega t\right) + \cos^2\psi \cos i \sin\omega t\cos\omega t - \sin^2\psi \cos i \sin\omega t \cos\omega t$$

$$= \tfrac{1}{2}\sin 2\psi\left(\cos^2 i \cos^2\omega t - \sin^2\omega t\right) + \cos 2\psi \cos i \sin\omega t \cos\omega t$$

$$M^{TT}_{ij} = \begin{bmatrix} \frac{1}{2}(u^2 - v^2) & uv \\ uv & \frac{1}{2}(v^2 - u^2) \end{bmatrix} \quad \text{in } \hat{u}, \hat{v} \text{ basis}$$

take two time derivatives

$$f = \cos \phi(t)$$

$$f' = -\dot{\phi} \sin \phi(t)$$

$$f'' = -\cancel{\ddot{\phi} \sin \phi} - \dot{\phi}^2 \cos \phi$$

suppose ϕ changing slowly

Ignore $\ddot{\phi}$ terms.

Set $\dot{\phi} = \omega$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\frac{d}{dt}(\cos^2 \omega t) = \frac{d}{dt}\left[\frac{1}{2}(1 + \cos 2\omega t)\right] = -\frac{1}{2}(2\omega)\sin(2\omega t) = -\omega(\sin 2\omega t)$$

$$\frac{d}{dt}(\sin^2 \omega t) = \omega \sin(2\omega t)$$

$$\frac{d}{dt}(2 \sin \omega t \cos \omega t) = \frac{d}{dt}\sin(2\omega t) = 2\omega \cos(2\omega t)$$

$$\frac{d^2}{dt^2}\cos^2 \omega t = -2\omega^2 \cos(2\omega t)$$

$$\frac{d^2}{dt^2}\sin^2 \omega t = 2\omega^2 \cos(2\omega t)$$

$$\frac{d^2}{dt^2}(2 \sin \omega t \cos \omega t) = -4\omega^2 \sin(2\omega t)$$

$$m^{TT}_{ij} = (\hat{u}_i\hat{u}_j - \hat{v}_i\hat{v}_j)\,\tfrac{1}{2}(u^2 - v^2) + (\hat{u}_i\hat{v}_j + \hat{v}_i\hat{u}_j)\,uv$$

$$m^{TT}_{ij} = (\hat{u}_i\hat{u}_j - \hat{v}_i\hat{v}_j)\,\tfrac{1}{2}\ddot{(u^2 - v^2)} + (\hat{u}_i\hat{v}_j + \hat{v}_i\hat{u}_j)\,\ddot{(uv)}$$

$$\ddot{(u^2 - v^2)} = \cos 2\psi\left[\cos^2 i\,(-2w^2\cos 2wt) - 2w^2\cos 2wt\right] - \sin 2\psi \cos i\,(-4w^2 \sin 2wt)$$

$$\ddot{(u^2 - v^2)} = -2w^2\cos 2\psi\,(\cos^2 i + 1)\cos(2wt) + 4w^2 \sin 2\psi \cos i \sin(2wt)$$

$$\ddot{(uv)} = -w^2 \sin 2\psi\,(\cos^2 i + 1)\cos(2wt) - 2w^2\cos 2\psi \cos i \sin(2wt)$$

Let $\alpha \equiv (\hat{u}_i\hat{u}_j - \hat{v}_i\hat{v}_j)(L^x_i L^x_j - L^y_i L^y_j)$

$\beta = (u_i v_j + v_i u_j)(L^x_i L^x_j - L^y_i L^y_j)$

Then
$$h(t) = \alpha w^2\left[-\cos 2\psi(\cos^2 i + 1)\cos 2wt + 2\sin 2\psi \cos i \sin 2wt\right]$$
$$+ \beta w^2\left[-\sin 2\psi(\cos^2 i + 1)\cos 2wt - 2\cos 2\psi \cos i \sin 2wt\right)$$

$$h(t) = \omega^2 \cos 2\omega t \,(\cos^2 i + 1)(-\alpha \cos 2\psi - \beta \sin 2\psi)$$
$$+ \omega^2 \sin 2\omega t \,\cos i \,(2\alpha \sin 2\psi - 2\beta \cos 2\psi)$$

$$= \omega^2 \left[X \sin(2\omega t) + Y \cos(2\omega t)\right]$$

$$X = 2 \cos i \,(\alpha \sin 2\psi - \beta \cos 2\psi)$$
$$Y = -(\cos^2 i + 1)(\alpha \cos 2\psi + \beta \sin 2\psi)$$

$$\alpha = (\hat{u}_i \hat{u}_j - \hat{v}_i \hat{v}_j)(L_x^i L_x^j - L_y^i L_y^j)$$
$$\beta = (\hat{v}_i \hat{u}_j + \hat{u}_i \hat{v}_j)(L_x^i L_x^j - L_y^i L_y^j)$$